Density in the calculus sequence

Martin Jackson

University of Puget Sound

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- multivariate calculus: primary motivation/interpretation for double, triple, line, and surface integrals (of scalar-valued functions)
- start with addressing the conception of density students bring to the calculus sequence

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  - from uniform to non-uniform
- start with a handout to introduce these generalizations in two steps

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5. Charge is distributed uniformly on a circular ring with a charge density of -4.21×10^{-6} Coulombs per cm. What is the total charge on a ring of radius 1.2 cm?

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volume density	$Q = \rho V$	$dQ = \rho \, dV$	$\longrightarrow$	$Q = \int \rho \, dV$

Calculus II project problem

Consider the problem of computing the total number of bacteria in a circular petri dish. The bacteria colony is more dense at the center than at the edges of the petri dish. Let r denote radial distance from the center of the dish measured in centimeters (cm). Let σ be the density of the bacteria colony, measured in number per square centimeter ($\#/\text{cm}^2$). Note that σ varies with radius r.

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(c) Get a numerical value for the total number with the density as in (b) and the values $\sigma_0 = 5.4 \times 10^3$ per cm^2 and $R = 5.5$ cm.

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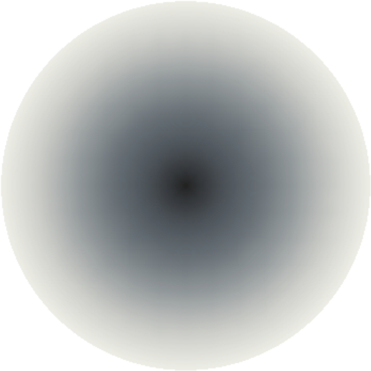

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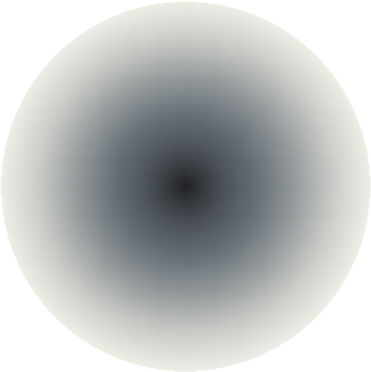

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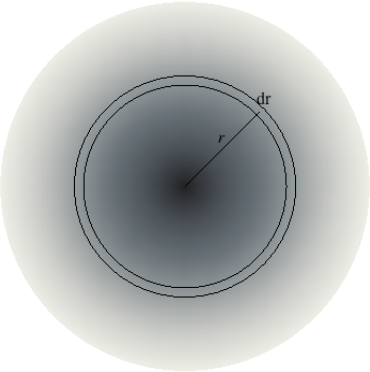

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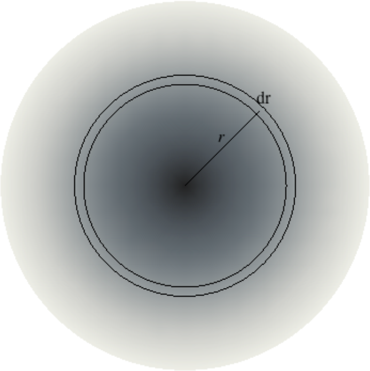

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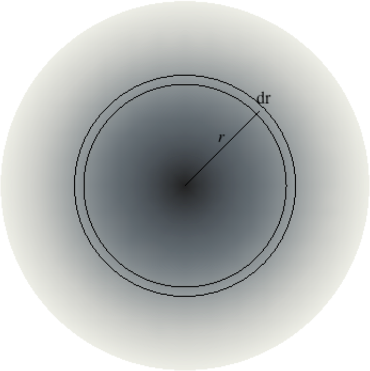

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summing: $m = \int_0^R 2\pi\sigma r\, dr$

Calculus II exam question

A rectangular piece of cloth is soaked in dye and then hung vertically to dry. As the cloth dries, the dye flows down so that more ends up at the bottom than at the top. The dried dye has a mass density that varies linearly from zero at the top edge to a maximum value at the bottom edge. Use H for the height of the cloth, W for the width of the cloth, and σ_0 for the maximum density.

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$$m = \int_0^H \sigma W\, dh = \int_0^H \left(\frac{\sigma_0}{H} h\right) W\, dh = \cdots = \frac{1}{2}\sigma_0 WH$$

Calculus III exam question

Charge is distributed on a hemisphere of radius R. Think of this as the northern hemisphere of the earth. The area charge density is proportional to the distance from the plane containing the equator with a value of 0 on the equator and a value of σ_0 at the north pole. Compute the total charge on the hemisphere in terms of R and σ_0.

Calculus III project problem

A hydrogen atom consists of one proton and one electron. A *free* hydrogen atom is one that experiences no external forces. In a free hydrogen atom, the electron can be in one of infinitely many discrete states. These states are labeled by three integers, usually denoted n, l, and m. For each state, there is an *electron location probability density* that gives the probability density (per volume) for the location of the electron as a function of position (measured with respect to the proton).

The $n = 3$, $l = 2$, $m = 0$ state of a free hydrogen atom has an electron probability density (per volume) given by

$$\rho(r, \phi, \theta) = \frac{1}{39366\pi} r^4 e^{-2r/3} (3\cos^2\phi - 1)^2$$

where (r, ϕ, θ) are spherical coordinates as we use them in class. The origin of the coordinate system is the location of the proton. The radial coordinate r is measured in units of *Bohr radii* where the Bohr radius is equal to about 5.3×10^{-11} meters. (So, for example, $r = 2$ means a radial distance of 2 Bohr radii.)

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3. Compute the probability of finding the electron anywhere in space. Does this result make sense?

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